

SOLUTION OPERATOR OF INHOMOGENUOUS DIRICHLET PROBLEM IN THE UNIT BALL

DAVID KALAJ AND DJORDJIJE VUJADINOVIĆ

ABSTRACT. In this paper we estimate norms of integral operator induced with Green function related to the Poisson equation in the unit ball with vanishing boundary data.

1. INTRODUCTION AND NOTATION

We denote by B^n and S^{n-1} the unit ball and unit sphere in R^n respectively. Throughout the paper we will assume that $n > 2$ (the case $n = 2$ has been already treated in [14, 15]). By the vector norm $|\cdot|$ we consider $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, and by the norm of an operator $T : X \rightarrow Y$ which acts between two normed spaces X and Y we mean

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$

Let P be the Poisson kernel, i.e. function

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n},$$

and let G be the Green function of the unit ball w.r.t. Laplace operator, i.e., the function

$$G(x, y) = c_n \left(\frac{1}{|x - y|^{n-2}} - \frac{1}{|x, y|^{n-2}} \right),$$

where

$$(1.1) \quad c_n = \frac{1}{(n-2)\omega_{n-1}},$$

where ω_{n-1} is the Hausdorff measure of S^{n-1} and

$$[x, y] := |x|y - y|y| = |y|x - x|x|.$$

As it is known, both functions P and G are harmonic for $|x| < 1$ with $x \neq y$.

Let $f : S^{n-1} \rightarrow R^n$ be a L^1 integrable function on the unit sphere S^{n-1} , and let $g : B^n \rightarrow R^n$ be L^1 integrable function in the unit ball. The solution

2000 *Mathematics Subject Classification.* Primary 35J05; Secondary 47G10.

Key words and phrases. Möbius transformations, Poisson equation, Newtonian potential, Cauchy transform, Bessel function.

of the Poisson equation $\Delta u = g$ (in the sense of distributions), in the unit ball, satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by (1.2)

$$u(x) = P[f](x) - \mathcal{G}[g](x) := \int_{S^{n-1}} P(x, \eta) f(\eta) d\sigma(\eta) - \int_{B^n} G(x, y) g(y) dy,$$

$|x| < 1$. Here $d\sigma$ is the normalized Lebesgue $n - 1$ dimensional measure of the Euclid sphere.

We consider the Poisson equation with inhomogenous Dirichlet boundary condition

$$(1.3) \quad \begin{cases} \Delta u(x) &= g, x \in B^n \\ u|_{\partial B^n} &= 0 \end{cases}$$

where $g \in L^p(B^n)$, $p \geq 1$. The weak solution is then given by

$$(1.4) \quad u(x) = -\mathcal{G}[g](x) = - \int_{B^n} G(x, y) g(y) dy, |x| < 1.$$

The main goal of our paper is related to estimating various norms of the integral operator \mathcal{G} . We call it the *solution operator* of Dirichlet's problem. The compressive study of this problem for $n = 2$ has been done by the first author in [14]. In [15] it is considered its counterpart for *differential operator* of Dirichlet's problem. For some related results concerning the planar case we refer to the papers [2, 4, 5, 6, 7]. In [3], Anderson, Khavinson and Lomonosov considered the L^2 norm of the operator

$$\mathcal{N}[f](x) =: \frac{1}{(n-2)\omega_{n-1}} \int_{B^n} \frac{1}{|x-y|^{n-2}} f(y) dy.$$

The following two results extend and generalize the corresponding results obtained in [14] and [3].

Theorem 1.1. *Let $\mathcal{G} : L^p(B) \rightarrow L^\infty(B)$, where $p > n/2$. Then*

$$\|\mathcal{G}\| = c_n \left(\frac{\pi^{n/2} \Gamma(1+q) \Gamma\left(\frac{n-q(-2+n)}{-2+n}\right)}{\Gamma\left(1+\frac{n}{2}\right) \Gamma\left(\frac{-n}{-2+n}\right)} \right)^{\frac{1}{q}}, \quad 1 \leq q < \frac{n}{n-2}$$

where $n \geq 3$ and $1/p + 1/q = 1$. In particular for $p = \infty$

$$\|\mathcal{G}\|_\infty = \frac{1}{2n} \quad (n \geq 3).$$

Remark 1.2. The particular case $p = \infty$ ($q = 1$) of Theorem 1.1, is simply and follows from the following observation. Since the function $u(x) = -\frac{1}{2n}(1 - |x|^2)$ represents unique solution of Poisson equation

$$\begin{cases} \Delta u(x) &= 1, x \in \Omega \\ u|_{\partial\Omega} &= 0 \end{cases},$$

it follows that for any integer $n, n \geq 3$ we have

$$(1.5) \quad \|\mathcal{G}\|_\infty = \sup_{x \in B^n} \left| \int_{B^n} G(x, y) dy \right| = \frac{1}{2n} \sup_{x \in B^n} (1 - |x|^2) = \frac{1}{2n}.$$

Theorem 1.3. *For $p \geq 1$, the operator \mathcal{G} is a bounded operator of the space L^p onto itself with the norm $\|\mathcal{G}\|_p$ satisfying the inequalities*

$$\|\mathcal{G}\|_p \leq (2n)^{\frac{p-2}{p}} \lambda_1^{\frac{2(1-p)}{p}}, \quad 1 \leq p \leq 2$$

and

$$\|\mathcal{G}\|_p \leq \lambda_1^{-\frac{2}{p}} (2n)^{\frac{p-2}{p}}, \quad 2 \leq p \leq \infty$$

which reduces to an equality for $p = 1, 2, \infty$, where $\lambda_1 = \lambda_1(B^n)$ is the first eigenvalues of Dirichlet Laplacian of the unit ball defined in Subsection 2.3.

The proof of Theorem 1.1 is postponed in section 4 and is obtained via Möbius transformations of the unit ball. It depends in Lemma 3.1, which is somehow very involved and presents itself a subtle integral inequality. The proof of Theorem 1.3, uses the eigenvalues of Dirichlet Laplacian and follows from Riesz-Thorin interpolation theorem.

2. PRELIMINARIES

2.1. Gauss hypergeometric function. Through the paper we will often use the properties of the hypergeometric functions. First of all, the hypergeometric function $F(a, b, c, t) = {}_2F_1(a, b; c; t)$ is defined by the series expansion

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} t^n, \quad \text{for } |t| < 1,$$

and by the continuation elsewhere. Here $(a)_n$ denotes shifted factorial, i.e. $(a)_n = a(a+1)\dots(a+n-1)$ and a is any real number.

The following identities will be used in the proof of the main results of this paper:

Euler's identity:

$$(2.1) \quad F(a, b; c; t) = (1-t^2)^{c-a-b} F(c-a, c-b; c; t), \quad \operatorname{Re}(c) > \operatorname{Re}(b) > 0,$$

Pfaff's identity:

$$(2.2) \quad F(a, b; c; t) = (1-t^2)^{-a} F(a, c-b; c; \frac{t}{t-1}), \quad \operatorname{Re}(c) > \operatorname{Re}(b) > 0,$$

Differentiation identity:

$$(2.3) \quad \frac{\partial}{\partial t} F(a, b; c; t) = \frac{ab}{c} F(a+1, b+1; c+1; t),$$

and Kummer's Quadratic Transformation

$$(2.4) \quad F\left(a, b; 2b; \frac{4t}{(1+t)^2}\right) = (1+t)^{2a} F\left(a, a + \frac{1}{2} - b; b + \frac{1}{2}; t^2\right),$$

where above identity is true for every t for which both series converge.

By using the Chebychev's inequality one can easily obtain the following inequality for Gamma function (see [8]).

Proposition 2.1. *Let m, p and k be real numbers with $m, p > 0$ and $p > k > -m$: If*

$$(2.5) \quad k(p - m - k) \geq 0 \ (\leq 0)$$

then we have

$$(2.6) \quad \Gamma(p)\Gamma(m) \geq (\leq) \Gamma(p - k)\Gamma(m + k).$$

2.2. Möbius transformations of the unit ball. The set of isometries of the hyperbolic unit ball B^n is a Kleinian subgroup of all Möbius transformations of the extended space $\overline{\mathbf{R}}^n$ onto itself denoted by $\mathbf{Conf}(\mathbf{B}^n) = \mathbf{Isom}(\mathbf{B}^n)$. We refer to the Ahlfors' book [1] for detailed survey to this class of important mappings. In general a Möbius transform $T_x : B^n \rightarrow B^n$ has the form

$$(2.7) \quad z = T_x y = \frac{(1 - |x|^2)(y - x) - |y - x|^2 x}{[x, y]^2}.$$

Then we have

$$(2.8) \quad |T_x y| = \left| \frac{x - y}{[x, y]} \right|.$$

If dy denotes the volume measure in the ball, because $y = T_{-x}z$ is a conformal mapping, in view of (2.8) we have

$$(2.9) \quad dy = \left(\frac{1 - |x|^2}{[z, -x]^2} \right)^n dz.$$

2.3. Eigenvalues of Dirichlet Laplacian. First of all, it is known that there exist an orthonormal basis of $L^2(B^n)$ consisting of eigenfunctions $(\varphi_n)_n$ of Dirichlet Laplacian

$$(2.10) \quad \begin{cases} -\Delta u = \lambda u, & z \in B^n \\ u|_{\partial B^n} = 0 \end{cases}$$

with corresponding eigenvalues $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \dots$. The functions φ_n are real valued.

It is well known that $\lambda_1(B^n)$ is given by the square of the first positive zero of the Bessel function $J_{(n-1)/2}(t)$ of the first kind of order $\alpha = (n-1)/2$:

$$(2.11) \quad J_\alpha(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{t}{2} \right)^{2m + \alpha}.$$

3. THE MAIN LEMMA

Lemma 3.1. *Let*

$$I(t) = (1 - t^2)^{n-q(n-2)} \int_0^1 \frac{(1 - r^{n-2})^q r^{n-q(n-2)-1}}{(1 - r^2 t^2)^{n-q(n-2)+1}} dr, \quad 0 \leq t < 1,$$

where $n \geq 3$ is a natural number and $1 < q < \frac{n}{n-2}$. Then the maximal value of function $I(t)$ is attained for $t = 0$, i.e.,

$$(3.1) \quad \begin{aligned} \max_{0 \leq t < 1} I(t) &= I(0) = \int_0^1 (1 - r^{n-2})^q r^{n-q(n-2)-1} dr \\ &= \frac{\Gamma(1+q)\Gamma\left(\frac{n-q(n-2)}{n-2}\right)}{(n-2)\Gamma\left(1+q+\frac{n-q(n-2)}{n-2}\right)} \end{aligned}$$

Proof. At the beginning we will observe the case $n > 3$. For $a = n - q(n - 2)$ we have $0 < a < 2$ and the next expansion

$$(3.2) \quad \begin{aligned} I(t) &= (1 - t^2)^a \int_0^1 \frac{(1 - r^{n-2})^{\frac{n-a}{n-2}} r^{a-1}}{(1 - r^2 t^2)^{a+1}} dr \\ &= (1 - t^2)^a \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)}{\Gamma(a+1)k!} t^{2k} \int_0^1 (1 - r^{n-2})^{\frac{n-a}{n-2}} r^{2k+a-1} dr \\ &= \frac{\Gamma(2 + \frac{2-a}{n-2})(1 - t^2)^a}{(n-2)\Gamma(a+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)\Gamma(\frac{a+2k}{n-2})}{\Gamma\left(\frac{2(k+n-1)}{n-2}\right)k!} t^{2k}. \end{aligned}$$

Assume that $n \geq 3$ and $k \geq 0$. Let

$$K = \frac{2k}{n-2}, \quad M = 2 + \frac{2k+a}{n-2}, \quad P = 2 + \frac{2}{n-2}.$$

From (2.6) we have

$$(3.3) \quad \Gamma(M)\Gamma(P) \leq \Gamma(M-K)\Gamma(P+K).$$

By using the formula $\Gamma(x+1) = x\Gamma(x)$ and (3.3), we have

$$\begin{aligned} \frac{\Gamma\left(\frac{a+2k}{n-2}\right)}{\Gamma\left(\frac{2(k+n-1)}{n-2}\right)} &= \frac{\Gamma(2 + \frac{a+2k}{n-2})}{\Gamma(2 + \frac{2+2k}{n-2})} \frac{1}{\left(\frac{a+2k}{n-2}\right)\left(\frac{a+2k}{n-2} + 1\right)} \\ &\leq \frac{\Gamma(2 + \frac{a}{n-2})}{\Gamma(2 + \frac{2}{n-2})} \frac{1}{\left(\frac{a+2k}{n-2}\right)\left(\frac{a+2k}{n-2} + 1\right)}. \end{aligned}$$

We obtain for $a \in (0, 2)$

$$\begin{aligned}
\frac{I(t)}{\frac{\Gamma(2+\frac{2-a}{n-2})}{n-2}} : \frac{\Gamma(2+\frac{a}{n-2})}{\Gamma(2+\frac{2}{n-2})} &\leq \frac{(1-t^2)^a}{\Gamma(a+1)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k+1)}{\Gamma(1+k)} \frac{t^{2k}}{\left(\frac{a+2k}{n-2}\right) \left(\frac{a+2k}{n-2} + 1\right)} \\
&= \frac{(n-2)(1-t^2)^a}{a} F\left(\frac{a}{2}, 1+a, \frac{2+a}{2}, t^2\right) \\
&\quad - \frac{(n-2)(1-t^2)^a}{(n+a-2)} F\left(1+a, \frac{1}{2}(n+a-2), \frac{a+n}{2}, t^2\right) \\
&= \frac{(n-2)}{a} F\left(1, -\frac{a}{2}, 1+\frac{a}{2}, t^2\right) \\
&\quad - \frac{(n-2)a}{a(n+a-2)} F\left(1, \frac{1}{2}(n-a-2), \frac{a+n}{2}, t^2\right) \\
&:= J(t).
\end{aligned}$$

The last expression for the function $J(t)$ was obtained by using the identity (2.4). Further we have

$$\begin{aligned}
(3.4) \quad \frac{\partial J(t)}{\partial t} &= \frac{-2t(n-2)}{a+2} F\left(2, \frac{2-a}{2}, 2+\frac{a}{2}, t^2\right) \\
&\quad - \frac{2t(n-2)(n-a-2)}{(n+a-2)(a+n)} F\left(2, 1+\frac{1}{2}(n-a-2), 1+\frac{a+n}{2}, t^2\right) \\
&< 0.
\end{aligned}$$

We conclude that the maximal value of the function $I(t)$ for $t = 0$ is attained.

In order to prove the special case $n = 3$, $1 < q < 3$, of Lemma 3.1, we should notice that

$$\begin{aligned}
(3.5) \quad \max_{0 \leq t < 1} I(t) &= \max_{0 \leq t < 1} (1-t^2)^{3-q} \int_0^1 \frac{(1-r)^q r^{2-q}}{(1-r^2 t^2)^{4-q}} dr \\
&\leq \max_{0 \leq t < 1} (1-t^2)^{3-q} \int_0^1 \frac{(1-r)^q r^{2-q}}{(1-r t^2)^{4-q}} dr.
\end{aligned}$$

Put

$$J(t) := (1-t^2)^{3-q} \int_0^1 \frac{(1-r)^q r^{2-q}}{(1-r t^2)^{4-q}} dr, 0 \leq t < 1.$$

By using the Taylor expansion we obtain

$$(3.6) \quad J(t) = \frac{\Gamma(1+q)\Gamma(3-q)}{6} (1-t^2)^{3-q} F(4-q, 3-q, 4; t^2), 0 \leq t < 1.$$

By using (2.1) and (2.2) respectively on expression for $J(t)$ we have

$$(3.7) \quad J(t) = \frac{\Gamma(1+q)\Gamma(3-q)}{6} F\left(q, 3-q, 4; \frac{t^2}{t^2-1}\right), 0 \leq t < 1.$$

So,

$$(3.8) \quad \begin{aligned} \max_{0 \leq t < 1} J(t) &= \frac{\Gamma(1+q)\Gamma(3-q)}{6} \max_{0 \leq t < 1} F\left(q, 3-q, 4; \frac{t^2}{t^2-1}\right) \\ &= \frac{\Gamma(1+q)\Gamma(3-q)}{6} \max_{0 \leq t < 1} F(q, 3-q, 4; 0). \end{aligned}$$

The last equality is a consequence of the fact that $\frac{t^2}{t^2-1} < 0$ and that coefficients

$$\frac{(q)_k(3-q)_k}{(1)_k(4)_k}$$

of the hypergeometric function

$$F\left(q, 3-q, 4; \frac{t^2}{t^2-1}\right)$$

are decreasing with respect to $k \geq 1$.

So,

$$(3.9) \quad \max_{0 \leq t < 1} I(t) = I(0) = \int_0^1 (1-r)^q r^{2-q} dr = \frac{\pi q(1-q)(2-q)}{6 \sin \pi q}.$$

□

4. PROOF OF THEOREM 1.1

We start this section with an easy lemma.

Lemma 4.1. *Let $\|\mathcal{G}\| := \|\mathcal{G} : L^p(B^n) \rightarrow L^\infty(B^n)\|$ for $p > \frac{n}{2}$. Then*

$$\|\mathcal{G}\| = \sup_{x \in B^n} \left(\int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Let $u(x) = \mathcal{G}[g](x)$, $g \in L^p(B)$. Hölder inequality implies

$$\|u\|_\infty \leq \sup_{x \in B} \left(\int_B |G(x, y)|^q dy \right)^{\frac{1}{q}} \left(\int_B |g(y)|^p dy \right)^{\frac{1}{p}},$$

i.e.,

$$\|\mathcal{G}\| \leq \sup_{x \in B^n} \left(\int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}}.$$

On the other hand, there exist $x_0 \in B^n$ so that

$$\left(\int_{B^n} |G(x_0, y)|^q dy \right)^{\frac{1}{q}} > \sup_{x \in B^n} \left(\int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}} - \epsilon.$$

We fix $x_0 \in B^n$. Let us consider the function

$$g(y) = \frac{(G(x_0, y))^{q-1}}{\|(G(x_0, y))^{q-1}\|_p}.$$

Then

$$\begin{aligned}
\|\mathcal{G}\| &\geq |\mathcal{G}[g](x_0)| \\
&= \left(\int_{B^n} |G(x_0, y)|^q dy \right)^{-\frac{1}{p}} \int_{B^n} |G(x_0, y)|^q dy \\
(4.1) \quad &= \left(\int_{B^n} |G(x_0, y)|^q dy \right)^{\frac{1}{q}} \\
&> \sup_{x \in B^n} \left(\int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}} - \epsilon,
\end{aligned}$$

i.e.,

$$\|\mathcal{G}\| = \sup_{x \in B^n} \left(\int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}}.$$

□

Proof of Theorem 1.1. We divide the proof into two cases.

(i) This case includes the following range for (n, q) : $n > 3$, with $1 < q < \frac{n}{n-2}$ and $n = 3$ with $q \in (2, 3)$. According to Lemma 4.1,

$$\|\mathcal{G}\| = \sup_{x \in B^n} \left(\int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}}, q > 1.$$

Further we have

$$(4.2) \quad \|\mathcal{G}\|^q = c_n^q \sup_{x \in B^n} \int_{B^n} \frac{1}{|x - y|^{q(n-2)}} \left| 1 - \left| \frac{x - y}{[x, y]} \right|^{n-2} \right|^q dy,$$

where c_n is defined in (1.1). We use the change of variable $z = T_x y$ i.e. $T_{-x} z = y$, in the previous integral where $T_x y$ Möbius transform defined in (2.7). By (2.9), denoting $t = |x|$, we obtain,

$$\begin{aligned}
&\sup_{x \in B^n} \int_{B^n} |G(x, y)|^q dy \\
&= \sup_{x \in B^n} c_n^q \int_{B^n} \frac{1}{|x - T_{-x} z|^{q(n-2)}} |1 - |z|^{n-2}|^q \frac{(1 - t^2)^n}{[z, -x]^{2n}} dz \\
&= c_n^q \sup_{x \in B^n} (1 - t^2)^n \int_{B^n} \frac{(1 - |z|^{n-2})^q}{\left| \frac{x[z, -x]^2 - (1-t^2)(x+z) - |x+z|^2 x}{[z, -x]^2} \right|^{q(n-2)}} \frac{dz}{[z, -x]^{2n}} \\
&= c_n^q \sup_{x \in B^n} (1 - t^2)^n \int_{B^n} \frac{(1 - |z|^{n-2})^q}{|z|^{q(n-2)} \left| \frac{1-t^2}{[z, -x]} \right|^{q(n-2)}} \frac{dz}{[z, -x]^{2n}}
\end{aligned}$$

$$\begin{aligned}
 &= c_n^q \sup_{x \in B^n} (1-t^2)^{n-q(n-2)} \int_{B^n} \left(\frac{1-|z|^{n-2}}{|z|^{n-2}} \right)^q [z, -x]^{q(n-2)-2n} dz \\
 &= c_n^q \sup_{x \in B^n} (1-t^2)^{n-q(n-2)} \int_0^1 \frac{(1-r^{n-2})^q}{r^{q(n-2)+1-n}} dr \int_S \frac{d\xi}{|rx + \xi|^{2n-q(n-2)}} \\
 &= c_n^q \sup_{x \in B^n} (1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^q}{r^{1-a}} dr \int_S \frac{d\xi}{(r^2 t^2 + 2rt\xi_1 + 1)^{\frac{n+a}{2}}} \\
 &= c_n^q C_n \sup_{x \in B^n} (1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^q}{r^{1-a}} dr \int_{-1}^1 \frac{(1-s^2)^{\frac{n-3}{2}}}{(r^2 t^2 + 2rts + 1)^{\frac{n+a}{2}}} ds,
 \end{aligned}$$

where

$$a = n - q(n-2), \quad C_n = \frac{\omega_{n-1} \Gamma(n-1)}{2^{n-2} \Gamma^2(\frac{n-2}{2})}$$

and in last two equalities it was assumed without loss of generality that $x = te_1, \xi = (\xi_1, \dots, \xi_n)$. If we take change of variable

$$\tau = \frac{1-s}{2}$$

in the previous integral we have

$$\begin{aligned}
 (4.3) \quad \|\mathcal{G}\|^q : c_n^q &= C_n \sup_{x \in B^n} (1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^q}{r^{1-a}} dr \int_{-1}^1 \frac{(1-s^2)^{\frac{n-3}{2}}}{(r^2 t^2 + 2rts + 1)^{\frac{n+a}{2}}} ds \\
 &= 2^{n-2} C_n \sup_{x \in B^n} (1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^q r^{a-1}}{(1+rt)^{n+a}} dr \int_0^1 \frac{\tau^{\frac{n-3}{2}} (1-\tau)^{\frac{n-3}{2}}}{(1 - \frac{4rt\tau}{(1+rt)^2})^{\frac{n+a}{2}}} d\tau.
 \end{aligned}$$

On the other hand, for fixed r we have $\frac{4rt}{(1+rt)^2} < 1$ and

$$\begin{aligned}
 (4.4) \quad &\int_0^1 \frac{\tau^{\frac{n-3}{2}} (1-\tau)^{\frac{n-3}{2}}}{(1 - \frac{4rt\tau}{(1+rt)^2})^{\frac{n+a}{2}}} d\tau \\
 &= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+k)}{k! \Gamma(\lambda)} \left(\frac{4rt}{(1+rt)^2} \right)^k \int_0^1 \tau^{k+\frac{n-3}{2}} (1-\tau)^{\frac{n-3}{2}} d\tau \\
 &= \Gamma\left(\frac{n-1}{2}\right) \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+k) \Gamma(k+\frac{n-3}{2}+1)}{k! \Gamma(\lambda) \Gamma(n-1+k)} \left(\frac{4rt}{(1+rt)^2} \right)^k \\
 &= \frac{\Gamma^2(\frac{n-1}{2})}{\Gamma(n-1)} F\left(\lambda, \frac{n-1}{2}; n-1; \frac{4rt}{(1+rt)^2}\right),
 \end{aligned}$$

where $\lambda = \frac{n+a}{2}$.

By using Kummer quadratic transformation and Euler's transformation for

hypergeometric functions, for $t = |x|$, we obtain

$$\begin{aligned}
 (4.5) \quad & \sup_{x \in B^n} (1-t^2)^a \int_0^1 \frac{(1-r^{n-2})^q r^{a-1}}{(1+rt)^{n+a}} F\left(\lambda, \frac{n-1}{2}; n-1; \frac{4rt}{(1+rt)^2}\right) dr \\
 &= \sup_{x \in B^n} (1-t^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1} F\left(\frac{n+a}{2}, \frac{a+2}{2}; \frac{n}{2}; r^2 t^2\right) dr \\
 &= \sup_{x \in B^n} (1-t^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1} (1-r^2 t^2)^{-a-1} \mathcal{F}(rt) dr \\
 &\leq \sup_{x \in B^n} (1-t^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1} (1-r^2 t^2)^{-a-1} \max_{t \leq 1} \mathcal{F}(rt) dr,
 \end{aligned}$$

where

$$\mathcal{F}(s) = F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; s^2\right).$$

Then by using the identity for the derivative of hypergeometric function we obtain

$$\begin{aligned}
 (4.6) \quad & \frac{\partial}{\partial t} F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; r^2 t^2\right) \\
 &= -2r^2 t \frac{2}{n} \frac{a}{2} \frac{q(n-2)-2}{2} F\left(\frac{q(n-2)-n+2}{2}, \frac{q(n-2)}{2}; \frac{n+2}{2}; r^2 t^2\right) < 0,
 \end{aligned}$$

for any $t \in [0, 1]$, which implies

$$(4.7) \quad \max_{|x| \leq 1} F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; r^2 |x|^2\right) = F\left(-\frac{a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; 0\right).$$

Finally, according to Lemma 3.1, for $n > 3$ the maximal value of the function

$$\begin{aligned}
 \mathcal{I}(x) &= \int_{B^n} |G(x, y)|^q dy \\
 &= c_n^q (1-|x|^2)^a \int_0^1 (1-r^{n-2})^q r^{a-1} dr \int_S \frac{d\xi}{|rx + \xi|^{2n-q(n-2)}}
 \end{aligned}$$

is attained for $x = 0$. So,

(4.8)

$$\begin{aligned}
\|G\|^q : c_n^q &= \sup_{x \in B^n} (1 - |x|^2)^a \int_0^1 (1 - r^{n-2})^q r^{a-1} dr \int_S \frac{d\xi}{|rx + \xi|^{n+a}} \\
&= \omega_{n-1} \sup (1 - |x|^2)^a \int_0^1 \frac{(1 - r^{n-2})^q r^{a-1}}{(1 - r^2|x|^2)^{a+1}} \mathcal{F}(r|x|) dr \\
&= \omega_{n-1} \int_0^1 (1 - r^{n-2})^q r^{n-q(n-2)-1} dr F\left(\frac{n+a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; 0\right) \\
&= \omega_{n-1} \int_0^1 (1 - r^{n-2})^q r^{a-1} dr = \frac{\omega_{n-1} \Gamma(1+q) \Gamma(\frac{n-q(n-2)}{n-2})}{(n-2) \Gamma(1+q + \frac{n-q(n-2)}{n-2})}.
\end{aligned}$$

(ii) The case $n = 3$ with $1 < q \leq 2$. It is clear that

$$\mathcal{I}(x) = \int_{B^3} |G(x, y)|^q dy = \frac{1}{(2\pi)^q} \int_{B^3} \left(\frac{1}{|x-y|} - \frac{1}{[x, y]} \right)^q dy,$$

and that the same transforms for $I(x)$ as in the previous general case give

$$\mathcal{I}(x) = c_3 (1 - x^2)^{3-q} \int_0^1 (1 - r)^q r^{2-q} F[(6-q)/2, (5-q)/2, 3/2, r^2 x^2] dr,$$

where c_3 is appropriate constant as in general case. Put $t = |x|$. We can represent $\mathcal{I}(x)$ as

$$\mathcal{I}(x) = c_3 \int_0^1 \frac{(1-r)^q r^{1-q} (1-t^2)^{3-q} ((1-rt)^{-4+q} - (1+rt)^{-4+q})}{2(4-q)t} dr.$$

So,

$$\mathcal{I}(x) = c_3 \frac{(1-t^2)^{3-q}}{2(4-q)t} \sum_{n=0}^{\infty} t^n \int_0^1 (1-r)^q r^{1-q} (r^n - (-r)^n) \binom{-4+q}{n} dr,$$

and this implies

$$\mathcal{I}(x) = c_3 \frac{(1-t^2)^{3-q}}{2(4-q)t} \sum_{n=0}^{\infty} \frac{(-1 + e^{in\pi}) \binom{-4+q}{n} \Gamma(2+n-q) \Gamma(1+q)}{\Gamma(3+n)} t^n.$$

Thus

$$\mathcal{I}(x) = c_3 \frac{\pi(-1+q)q (1-t^2)^{3-q} (F(2-q, 4-q; 3; t) - F(2-q, 4-q; 3; -t))}{4 \sin(\pi q)(4-q)t}.$$

Let

$$c'(q) := c_3 \frac{2^{-q} \pi^{2-q} (-1+q)q}{(4-q) \sin(\pi q)}.$$

Then

$$\mathcal{I}(x)/|c'| \leq I_1(x) = \frac{(1-t^2)}{t} (F(2-q, 4-q; 3; t) - F(2-q, 4-q; 3; -t))$$

for $1 < q < 2$ and

$$I_1(x) = a_0 + \sum_{n=1}^{\infty} a_n t^n$$

where $a_0 > 0$ and

$$a_n = \frac{2(1 + (-1)^n) \Gamma(3 + n - q) (-(n - q)! \Gamma(4 + n) + \Gamma(n) \Gamma(5 + n - q))}{\Gamma(n) \Gamma(2 + n) \Gamma(4 + n) \Gamma(2 - q) \Gamma(4 - q)}.$$

Further $a_n \leq 0$ because

$$\frac{(1 + n - q)(2 + n - q)(3 + n - q)(4 + n - q)}{n(1 + n)(2 + n)(3 + n)} \leq 1,$$

which again implies that maximal value of the function $\mathcal{I}(x)$ is attained for the $x = 0$. This finishes the proof of Theorem 1.1. \square

5. PROOF OF THEOREM 1.3

Let Ω be a domain of \mathbf{R}^n and let $|\Omega|$ be its volume. For $\mu \in (0, 1]$ define the operator V_μ on the space $L^1(\Omega)$ by Riesz potential

$$(V_\mu f)(x) = \int_{\Omega} |x - y|^{n(\mu-1)} f(y) dy.$$

The operator V_μ is defined for any $f \in L^1(\Omega)$ and V_μ is bounded on $L^1(\Omega)$, or more generally we have the next lemma.

Lemma 5.1. [11, p. 156-159]. *Let V_μ be defined on the $L^p(\Omega)$, $p > 0$. Then V_μ is continuous as a mapping $V_\mu : L^p(\Omega) \rightarrow L^q(\Omega)$, where $1 \leq q \leq \infty$, and*

$$0 \leq \delta = \delta(p, q) = \frac{1}{p} - \frac{1}{q} < \mu.$$

Moreover, for any $f \in L^p(\Omega)$

$$\|V_\mu f\|_q \leq \left(\frac{1 - \delta}{\mu - \delta} \right)^{1-\delta} (\omega_{n-1}/n)^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p.$$

Remark 5.2. If instead of Riesz potential we consider the solution operator for a domain Ω with finite volume, then the operator is in general non-bounded. However it is bounded, if the boundary of Ω is enough regular. See [12] for an essential approach to the solution of this problem.

Theorem 5.3. *Let $\|\mathcal{G}\|_1 := \|\mathcal{G} : L^1(B) \rightarrow L^1(B)\|$, then*

$$\|\mathcal{G}\|_1 = \frac{1}{2n}.$$

Proof. According to Theorem 1.1 we have

$$\|\mathcal{G}\|_{L^\infty \rightarrow L^\infty} = \frac{1}{2n}.$$

On the other hand, Lemma 5.1 states that $\mathcal{G} : L^1 \rightarrow L^1$ is bounded. Then

$$\|\mathcal{G}\|_{L^1 \rightarrow L^1} = \|\mathcal{G}^*\|_{L^\infty \rightarrow L^\infty},$$

where \mathcal{G}^* is appropriate adjoint operator. Since

$$\mathcal{G}^* f(x) = \int_{B^n} \overline{G(y, x)} f(y) dy = \int_{B^n} G(x, y) f(y) dy, f \in L^\infty(B),$$

we have

$$\|\mathcal{G}\|_{L^1 \rightarrow L^1} = \|\mathcal{G}\|_{L^\infty \rightarrow L^\infty}.$$

□

In the sequel we are going to observe Hilbert case $p = 2$, $\mathcal{G} : L^2(B) \rightarrow L^2(B)$. It is well-known that $\mathcal{G}^{-1} = -\Delta$ on the Sobolev space $H_0^1(\Omega)$, so the Hilbert norm \mathcal{G} is precisely the reciprocal value of the norm of $-\Delta$ (c.f. [3]). So we have the following theorem, whose proof is included for the sake of completeness.

Theorem 5.4. *Let $\|\mathcal{G}\|_2 := \|\mathcal{G} : L^2(B^n) \rightarrow L^2(B^n)\|$, then*

$$\|\mathcal{G}\|_2 = \frac{1}{\lambda_1}.$$

Thus

$$(5.1) \quad \|\mathcal{G}g\|_2 \leq \frac{1}{\lambda_1} \|g\|_2, \quad g \in L^2(B^n).$$

Equality is attained in (5.1) for $g(x) = c\varphi_1(x)$, a.e. $x \in B^n$ where c is a real constant.

Proof. If $f \in L^2(B^n)$, then under the previous notation

$$f(x) = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \varphi_k(x).$$

Since \mathcal{G} is bounded, we have

$$\mathcal{G}[f] = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \mathcal{G}[\varphi_k].$$

Also,

$$\mathcal{G}[\varphi_k] = \frac{1}{\lambda_k} \mathcal{G}[\Delta \varphi_k] = -\frac{1}{\lambda_k} \varphi_k.$$

The fact that (φ_k) is orthonormal implies

$$\|\mathcal{G}f\|_2^2 = \sum_{k=1}^{\infty} \frac{|\langle f, \varphi_k \rangle|^2}{\lambda_k^2}.$$

Since λ_1 is a simple eigenvalue and $0 < \lambda_1 < \lambda_2 \leq \dots$, we have

$$\|\mathcal{G}f\|_2 \leq \frac{1}{\lambda_1} \|f\|_2.$$

Finally,

$$\|\mathcal{G}\|_2 = \frac{1}{\lambda_1}.$$

□

By using the Riesz-Thorin interpolation theorem [17], we obtain the following estimates of the norm of the operator $\mathcal{G} : L^p \rightarrow L^p$.

Let us denote by $\|\mathcal{G}\|_{L^1 \rightarrow L^1} = \|\mathcal{G}\|_{L^\infty \rightarrow L^\infty} = \|\mathcal{G}\|_1$ and $\|\mathcal{G}\|_{L^2 \rightarrow L^2} = \|\mathcal{G}\|_2$. Then

$$\|\mathcal{G}\|_p \leq \|\mathcal{G}\|_1^{\frac{2-p}{p}} \|\mathcal{G}\|_2^{\frac{2(p-1)}{p}} = (2n)^{\frac{p-2}{p}} \lambda_1^{\frac{2(1-p)}{p}},$$

where $\|\mathcal{G}\|_p$ represents the norm of the operator $\mathcal{G} : L^p(B^n) \rightarrow L^p(B^n)$, $1 < p < 2$. Similarly,

$$\|\mathcal{G}\|_p \leq \|\mathcal{G}\|_2^{\frac{2}{p}} \|\mathcal{G}\|_1^{\frac{p-2}{p}} = \lambda_1^{-\frac{2}{p}} (2n)^{\frac{p-2}{p}},$$

where $\mathcal{G} : L^p(B^n) \rightarrow L^p(B^n)$, $2 < p < \infty$. This yields the proof of Theorem 1.3.

REFERENCES

- [1] L. V. Ahlfors: *Möbius transformations in several dimensions* University of Minnesota, School of Mathematics, 1981, 150 p.
- [2] J. M. Anderson, A. Hinkkanen: *The Cauchy transform on bounded domains*. Proc. Amer. Math. Soc. **107** (1989), no. 1, 179–185.
- [3] J. M. Anderson, D. Khavinson; V. Lomonosov: *Spectral properties of some integral operators arising in potential theory*. Quart. J. Math. Oxford Ser. (2) **43** (1992), no. 172, 387–407.
- [4] A. Baranov; H. Hedenmalm: *Boundary properties of Green functions in the plane* Duke Math. J. **145** (2008), no. 1, 1–24.
- [5] M. Dostanić: *Norm estimate of the Cauchy transform on $L^p(\Omega)$* . Integral Equations Operator Theory **52** (2005), no. 4, 465–475.
- [6] M. Dostanić: *Estimate of the second term in the spectral asymptotic of Cauchy transform*. J. Funct. Anal. **249** (2007), no. 1, 55–74.
- [7] M. Dostanić: *The properties of the Cauchy transform on a bounded domain*, Journal of the Operator Theory **36** (1996), 233–247
- [8] S.S. Dragomir, R.P. Agarwal, N. S. Barnett: *Inequalities for Beta and Gamma functions via some classical and new integral inequalities*. (English) J. Inequal. Appl. **5**, No.2, 103–165 (2000).
- [9] H. Hedenmalm, S. Shimorin: *Weighted Bergman spaces and the integral means spectrum of conformal mappings*. Duke Math. J. **127** (2005), no. 2, 341–393.
- [10] R. Durán; M. Sanmartino; M. Toschi: *Weighted a priori estimates for the Poisson equation*, Indiana Univ. Math. J. **57** (2008), no. 7, 3463–3478.
- [11] D. Gilbarg and N. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer-Verlag, Berlin, 1983. xiii+513 pp.
- [12] D. Jerison, C. E. Kenig: *The inhomogeneous Dirichlet problem in Lipschitz domains*. J. Funct. Anal. **130** (1995), no. 1, 161–219.
- [13] J. Jost: *Compact Riemann surfaces. An introduction to contemporary mathematics*. Third edition. Universitext. Springer-Verlag, Berlin, 2006. xviii+277 pp.
- [14] D. Kalaj: *On Some Integral Operators Related to the Poisson Equation*, Integral Equation and Operator Theory **72** (2012), 563–575.
- [15] D. Kalaj: *Cauchy transform and Poisson’s equation*, Advances in Mathematics Volume 231, Issue 1, 10 September 2012, Pages 213–242.
- [16] D. Kalaj, M. Pavlović: *On quasiconformal self-mappings of the unit disk satisfying the Poisson’s equation*, Trans. Amer. Math. Soc. **363** (2011) 4043–4061.

- [17] G. Thorin: *Convexity theorems generalizing those of M. Riesz and Hadamard with some applications.* Comm. Sem. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **9**, (1948), 1–58.
- [18] X. Tolsa: *L^2 -boundedness of the Cauchy integral operator for continuous measures.* Duke Math. J. 98 (1999), no. 2, 269–304.

UNIVERSITY OF MONTENEGRO, FACULTY OF MATHEMATICS, DZORDZA VAŠINGTONA
BB, 81000 PODGORICA, MONTENEGRO
E-mail address: `djordjijevuj@t-com.me`